A Rational Approach to B-H Curve Representation

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The paper introduces an approach to constructing a rational function that fits the finite set of data points on the B-H plane representing the non-linear response of a permeable material, providing an approximation to the data well-suited for interpolation and extrapolation. Improving on previous methods, this approach provides a smooth, closed-form approximation to the data capable of representing the material's response from its Rayleigh region through to its saturation, appropriate for use with finite element solvers. This is achieved by applying the method of vector fitting (as seen in the discipline of control systems) to the B-H dataset, while taking care to remove pole-zero pairs that may occur between data points. The method is demonstrated to provide high-accuracy approximations to the datasets of a range of materials.

***Index Terms*—Magnetic materials, magnetization, nonlinear magnetics, permeability**

I. Introduction

ALTHOUGH APPROXIMATIONS of non-linear permeability have been used in computational electromagnetics for over 40 years, high-performance materials and the need for greater simulation accuracy present challenges to current models. With application to the finite element method, such models usually take the form of continuously differentiable functions mapping magnetic field intensity, H, to magnetic flux density, B, for scalar-potential problems, or functions mapping B to H for vector-potential problems. Current techniques for representing B-H datasets with continuous functions can be classified into two categories: those using functions in a small, fixed number of parameters that are smooth over $\mathbb{R}_{\geq 0}$, and those using functions in a (comparatively) large number of parameters that are piecewise-defined over a finite interval of $\mathbb{R}_{\geq 0}$.

Examples of functions from the first category are found in [1]-[4], and include exponential decays, trigonometric arctangents, hyperbolic tangents, and first- and second-order rationals. These functions have the desirable characteristic of both interpolating and extrapolating the B-H dataset. However, by virtue of being defined by as few as two or three parameters, they trade off the ability to capture nuances - such as those resulting from the Rayleigh law as well as those present in modern, high-permeability materials - in favour of assured monotonicity and smoothness.

The second category of functions consists largely of piecewise-polynomial approximations (as in [5]-[7]), and interpolations (as in [8]). Often, these functions' parametrisations have degrees of freedom on the order of the size of the dataset they are intended to fit, and consequently provide a closer approximation than functions of the first category. However, as they are piecewise-defined, they do not have the same smoothness nor inherently allow for extrapolation of the B-H curve beyond the last point in the dataset. In [5], the extrapolation is achieved via a linear continuation of the piecewise function, with no assurance of differentiability at the start of the region. Extrapolations by rational function in [7] and by trigonometric arctangent in [8] both provide continuous differentiability over $\mathbb{R}_{\geq 0}$, but arguably are not representative of the saturation region of the material, as (in order to ensure differentiability) their construction is based solely on the last piecewise segment's value and derivative.

The approach presented here falls into the first category, and attempts to mitigate the shortcomings of previous methods by using high-order rational functions. The resulting functions map H onto B; their parameter values are determined using an iterative linear least-squares algorithm [9], with a final step of removing real pole-zero pairs that may occur between data points during the fitting process. The interpolation and extrapolation of the dataset are determined concurrently, producing smooth functions that well-approximate the input data points, and thus the material's response as a whole.

II. Theory

A. Iterative linear least-squares fit

Since it is known that the B-H curve asymptotically approaches a slope of μ_0 and passes through the point $B = H = 0$, a rational function of degree d of the form

$$\begin{aligned} f(H) &= B(H) - \mu_0 H \\ &= p(H)/q(H) \\ &= \left(\sum_{i=1}^{d} p_i H^i\right) \Big/ \left(1 + \sum_{j=1}^{d} q_j H^j\right) \end{aligned} \tag{1}$$

is fit to the dataset of m points $(H_k, B_k - \mu_0 H_k)$ for $k \in \{1, ..., m\}$. In [9], the weighted least-squares error function is chosen to be linear in the coefficients p_i and q_j by taking the form:

$$\sum_{k=1}^{m} \left(p\left(H_k\right) - \mu_0 M_k q\left(H_k\right)\right)^2 \tag{2}$$

where $\mu_0 M_k = B_k - \mu_0 H_k$. Unfortunately, (2) does not apply equal weight to each data point: a weight of $q\left(H_k\right)$ is applied to the data point $(H_k, \mu_0 M_k)$. To mitigate this, the error function is iteratively minimized in its parameters, and reweighted at

each step by dividing by the previous iteration's denominator, thus taking the form:

$$\epsilon^{(n)}\left(p_1^{(n)},...,p_d^{(n)},q_1^{(n)},...,q_d^{(n)}\right) = \sum_{k=1}^{m}\left(\frac{p^{(n)}(H_k)-\mu_0 M_k q^{(n)}(H_k)}{q^{(n-1)}(H_k)}\right)^2 \quad (3)$$

where parenthesised superscripts denote iteration count, and for the first step, $n = 1$, $q^{(0)}(H) = 1$. This results in an $m \times 2d$ overdetermined system of linear equations of the form $A^{(n)}\vec{x}^{(n)} \approx \vec{b}^{(n)}$,

$$A_{ij}^{(n)} = \begin{cases} H_i^j/q^{(n-1)}(H_i) & 1 \le j \le d \\ -\mu_0 M_i H_i^{j-d}/q^{(n-1)}(H_i) & d < j \le 2d \end{cases}$$
$$\vec{x}_i^{(n)} = \begin{cases} p_i^{(n)} & 1 \le i \le d \\ q_{i-d}^{(n)} & d < i \le 2d \end{cases} \quad (4)$$
$$\vec{b}_i^{(n)} = \mu_0 M_i/q^{(n-1)}(H_i)$$

whose best fit solution is achieved by the usual means of solving $A^T A\vec{x} = A^T\vec{b}$ for $\vec{x}$ at each step via QR decomposition for the sake of numerical stability.

This iteration is not assured to converge to a global minimum, and is terminated when the error function, $\epsilon^{(n)}$, achieves its first minimum (i.e. the iteration n for which $\epsilon^{(n)} < \epsilon^{(n+1)}$). Experimentation demonstrates that its convergence properties are favourable when compared with those of a Newton-Raphson treatment of the non-linear problem arising from a direct, least-squares fit of $f(H)$ to the data, as the latter is much more sensitive to the initial choice of p_i and q_j due to the fact that, for a rational of degree d with m data points, it is expected by Bézout's theorem that the non-linear least-squares error function has as many as $(4m-1)^{2d}$ local minima.

B. Pole-zero cancellation

The algorithm presented does not constrain the function $f(H)$ to be pole-free between data points. However, when a pole occurs in an interval spanning two data points, there is expected to be a zero in the same interval; experimentally, exceptions are extremely rare. The removal of pole-zero pairs has a minimal impact on the quality of the resulting fit due to their proximity to one another. They are identified by factoring the polynomials of the numerator and denominator via the eigenvalue decomposition of their companion matrices [10]. Poles and zeros on the positive real axis are then removed by polynomial long division of $q(H)$ and $p(H)$, respectively.

Noteworthy is the fact that removal of these pole-zero pairs reduces the order of the resulting rational function. As their number cannot be predetermined, it may be the case that these pairs "consume" degrees of freedom otherwise needed to accurately fit the original dataset. In this case, the algorithm is restarted with a larger value of d.

III. Results

The algorithm is applied to the Cobalt Steel - Hiperco 50, Stainless Steel - 419, and Castings - Cast Iron B-H datasets available from [11], constructing rational functions of initial degree $d = 5$ (empirically, this provides sufficient degrees of freedom to fit most datasets). As seen in the linear-log plot in Fig. 1, the algorithm produces a good fit over the entirety of each dataset, and provides reasonable values for saturation magnetization. The final degrees of the rational functions are 5, 5, and 3, respectively, as the cast iron dataset's iterative fit results in two pole-zero pairs. The root-mean-squared errors of the approximations are 1.5, 4.5, and 30.7 mT, respectively.

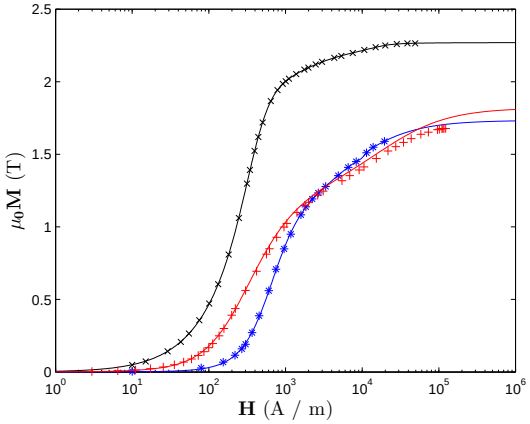

Fig. 1. Magnetization (multiplied by μ_0) versus magnetic field intensity for three sample materials: Cobalt Steel - Hiperco 50 dataset ('x' markers), Stainless Steel - 419 dataset ('*' markers), Castings - Cast Iron dataset ('+' markers), and their respective rational approximations (smooth lines).

This approach has been used to represent over 100 datasets by rational functions of degree 5 or lower for which no *a posteriori* corrections were required to ensure monotonicity and pole-free extrapolation.

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